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INFIMA OF ENERGY FUNCTIONALS IN HOMOTOPY CLASSES OF MAPPINGS

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0. Introduction

Let M and N be compact connected Riemannian manifolds. The energy of a lipschitz map $f\colon M \to N$ is $\int_M |Df|^2$ (where $|Df(x)|^2 = \sum |\partial f/\partial x_i|^2$ if $x_1, \cdots, x_m$ are normal coordinates for M at x). Mappings for which the first variation of energy vanishes are called harmonic. ([1], [5], and [7] are nice introductions to harmonic maps.) The identity map from M to M is always harmonic, but it may be homotopic to mappings of less energy. For instance the identity map on S^3 is homotopic to mappings of arbitrarily small energy (namely, conformal maps that pull points from the North Pole toward the South Pole). That suggests the question: For which manifolds M is the identity map homotopic to maps of arbitrarily small energy? In this paper we give the simple answer: Those M such that $\pi_1(M)$ and $\pi_2(M)$ are both trivial. More generally we consider energy functionals like $\Phi(f) = \int_M |Df|^p$ and ask: When is the infimum of $\Phi(f)$ in some homotopy class of mappings $f\colon M \to N$ equal to 0?

To answer such questions, it is convenient to regard N as isometrically embedded in a euclidean space $\mathbf{R}^\nu$ and to work with the Sobolev norm

$$\|f\|_{1,p} = \left(\int_M |f|^p\right)^{1/p} + \left(\int_M |Df|^p\right)^{1/p}$$

(where $f\colon M \to \mathbf{R}^\nu$ has distribution derivative Df) and with the associated Sobolev space

$$W^{1,p}(M, N) = \text{the } \|\cdot\|_{1,p} \text{ completion of } \{\text{lipschitz maps } f\colon M \to N\}$$

We say that two continuous maps f, $g\colon M \to N$ are k-homotopic (or have the same k-homotopy type) if their restrictions to the k-dimensional skeleton of some triangulation of M are homotopic. Then the gist of our main theorems (Theorems 1 and 2) can be summarized as follows.

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