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SURFACES IN 3-SPACE AND THEIR CONTACT WITH CIRCLES

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Two hundred years ago Meusnier established that for any surface in $\mathbf{R}^3$ the set of osculating circles at a point and with a given tangent direction form a sphere [14]. (An osculating circle is one with at least 3-point contact with the surface.)

In this paper we investigate higher-order contact between circles and generic surfaces using a singularity theory approach. This approach, which developed from an observation by Thom, is by now well established. See for example [1], [2], [4], [11] and [12]. The general idea is as follows. Let M be a parametrized family of 'model' submanifolds of $\mathbf{R}^n$ (spheres, circles, lines or whatever), and for each model submanifold m let $f_m\colon \mathbf{R}^n \to \mathbf{R}^p$ be a map which cuts out m. We require the map $F\colon \mathbf{R}^n \times M \to \mathbf{R}^p$, $F(y, m) = f_m(y)$, to be smooth. Let $g\colon X \hookrightarrow \mathbf{R}^n$ be an immersion of the manifold X. Consider the map

$$(1) \qquad \Phi\colon X \times M \to \mathbf{R}^p, \qquad \Phi(x, m) = \varphi_m(x) = f_m \circ g(x).$$

The contact of m and $g(X)$ (which we henceforth refer to as X) at $g(x)$ is then determined by the singularity type (more precisely, the $\mathcal{K}$-class) of the map φ_m at x [9]. The techniques and results of singularity theory can be used to recognize the contact types that occur in a given setting, and for 'generic' immersions of X both to predict which contact types can be expected and to give some global or semi-global information.

The main results of this article are contained in 5 theorems. Theorems 1 and 2 deal with the contact types that arise at each point of the surface. Theorems 3 and 4 deal with the behavior of circles with 6-point contact with the surface near umbilics. Theorem 5 is a generalization of a theorem of Banchoff, Gaffney and McCrory [2]. The genericity theorem needed for the later theorems is stated here, but is proved in greater generality in [10].

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