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THE SPECTRAL GEOMETRY OF A TOWER OF COVERINGS

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Let M be a compact Riemannian manifold and let $\{M_i\}$ be a family of finite-sheeted covering spaces of M with the induced Riemannian metric.

In this paper, we wish to study the behavior of the first eigenvalue $\lambda_1(M_i)$ as i tends to infinity. Here λ_1 is given by the variational formula

$$\lambda_1(N_i) = \inf_f \frac{\int_N \|df\|^2}{\int_N |f|^2}, \tag{1}$$

where f ranges over functions which are perpendicular to the constant function $\sim \int_N f = 0$.

It might appear, particularly from the perpendicularity condition, that the behavior of λ_1 depends rather delicately on the metric properties of M. However, motivated by our work on λ_0 of coverings [1], we were led to the point of view that the asymptotic properties of λ_1 as i tends to infinity should be governed only by combinatorial properties of the fundamental group of M.

Our main result, Theorem 1, confirms that this is indeed the case. To state the combinatorial property which emerges, let us first recall that a finite-sheeted covering space M_i of M is described by a subgroup $\pi_1(M_i)$ of finite index in $\pi_1(M)$. We now fix, once and for all, generators $g_1, \cdots, g_k$, and for each i we consider the combinatorial graph Γ_i described as follows: The vertices of Γ_i are the finite number of cosets of $\pi_1(M)/\pi_1(M_i)$. Two vertices of Γ_i are joined by an edge if the corresponding cosets differ by left multiplication by one of the g_i's.

For each i, we let h_i denote the following number: Let $E = \{E_j\}$ be a collection of edges of Γ_i such that $\Gamma_i - E$ disconnects into two pieces, A and B. Let $\#(E)$ denote the number of edges in E, and $\#(A)$ and $\#(B)$ the

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