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THE HEAT EQUATION SHRINKING CONVEX PLANE CURVES

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1.

Let M and M' be Riemannian manifolds and F: $M \to M'$ a smooth map. The Laplacian of F is defined intrinsically as a section of the pull-back of the tangent bundle of M,

$$\Delta F \in \mathscr{C}^{\infty}(M, F^{*}TM'),$$

and is given by the trace of the vector-valued matrix of second derivatives in any two systems of normal coordinates on M and M'. When F is an isometric immersion, the Laplacian of F is given by $\Delta F = kN$, where k is the mean curvature (the trace of the second fundamental form) and N is the unit normal vector. We can deform the immersion F by the heat equation

$$\frac{\partial F}{\partial t} = \Delta F \quad \text{or} \quad \frac{\partial F}{\partial t} kN$$

always computing ΔF in the varying metric on M induced by the immersion F. It is a theorem (see [5]) that the solution always exists for a short time, and is unique and smooth. Moreover, the immersed submanifolds $M_t = F_t(M)$ are independent of the parametrization. If two immersions F and F^* differ by a diffeomorphism h at time $t = 0$, then the solutions continue to satisfy $F^* = F \circ h$ as long as they exist for the same fixed h independent of t.

The equation is clearly of significant geometrical interest. It has the following variational interpretation. The space $\mathscr{M}$ of all immersed submanifolds M of M' has the structure of an infinite-dimensional manifold modeled on a Fréchet space (see [4]). the tangent space $T_M\mathscr{M}$ to $\mathscr{M}$ at M is naturally identified with the space $C^{\infty}(M)$ of functions f on M, where the variation in M is given by moving infinitesimally a distance f in the normal direction. The

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