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NORMAL BUNDLES FOR AN EMBEDDED $\mathbf{R}P^2$ IN A POSITIVE DEFINITE 4-MANIFOLD

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1. Introduction

In this paper we wish to study which normal bundles can occur for differentiable embeddings of the real projective plane $\mathbf{R}P^2$ into a positive definite 4-manifold. The techniques we use were developed by Donaldson [4] to show that only the standard intersection forms can occur for simply connected positive definite 4-manifolds and modified by Fintushel and Stern [5], [6] to reprove part of Donaldson's theorem as well as give results concerning homology 3-spheres and other applications. The starting point for our work is [6]; in particular, the application of their techniques to reprove Kuga's theorem was the main motivation for our approach. Kuga's theorem [13] gives restrictions on which homology classes $S^2 \times S^2$ can be realized by an embedded 2-sphere.

The Fintushel-Stern proof of Kuga's theorem goes roughly as follows. By cutting out a neighborhood of an embedded S^2 realizing a given homology class, one gets a positive definite 4-manifold X with $H^2(X) = \mathbf{Z}$ and boundary a certain lens space. Moreover, there is an $SO(2)$-bundle E over X which restricts to a specified bundle over the lens space so that one can construct a pseudofree orbifold and apply their general theory. However, the crucial point of the construction is not the pseudofree orbifold but rather the bundles involved. Also, their construction utilizes a branched covering which is not essential—it may be replaced by just forming a V-manifold $X \cup c\partial X$, where $c\partial X$ is covered by D^4. The key data needed is the manifold X with ∂X a lens space which is covered by S^3, $H^2(X) \approx \mathbf{Z}$, $H_1(X) = 0$, $X \cup c\partial X$ a positive definite rational homology manifold. Given that data, one can form a V-manifold using X and D^4, where D^4 covers the cone on the lens space, and form an $SO(2)$-bundle E over X corresponding to an Euler class $e \in H^2(X)$.

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