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FOLIATED CR MANIFOLDS

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Abstract

Let X be a (maximally complex) abstract CR manifold of dimension $4m + 1$ with nondegenerate Levi form. Suppose that X is foliated by compact complex manifolds of complex dimension $\geqslant m$. Then $m = 1$, the leaves are Riemann spheres, and X arises from a twistor construction.

Introduction

Consider for a moment the Hopf map

$$\pi\colon \mathbf{CP}_3 \to \mathbf{HP}_1 \approx S^4$$

obtained by remembering that a pair of quaternions is also a quadruple of complex numbers:

$$\pi([z_0, z_1, z_2, z_3]) = [z_0 + z_1 j, z_2 + z_3 j].$$

The inverse image of any point $x \in S^4$ is then a complex projective line $\mathbf{CP}_1 \subset \mathbf{CP}_3$, and if we take the inverse image $X = \pi^{-1}(M)$ of a hypersurface $M^3 \subset S^4$, we obtain a real hypersurface in $\mathbf{CP}_3$ foliated by compact complex curves. For example, if M is the equator $S^3 \subset S^4$ given by $\{[q_0, q_1] \in \mathbf{HP}_1 \mid \|q_0\| = \|q_1\|\}$, the $\mathbf{CP}_1$-foliated real hypersurface X is the real hyperquadric $\{[z_0, z_1, z_2, z_3] \in \mathbf{CP}_3 \mid |z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2 = 0\}$. The Levi form (§0) of this hyperquadric is clearly nondegenerate, a fact which carries over for *any* choice of $M^3 \subset S^4$.

How general is the above family of examples? For instance, is it possible to find a real hypersurface in a (possibly noncompact) complex 3-manifold which has nondegenerate Levi form and is foliated by compact holomorphic curves of higher genus? The answer, as shall be shown herein, is *no*.

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