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PLANES WITHOUT CONJUGATE POINTS

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1. Problems involving perturbations of canonical Riemannian metrics arise frequently. If one requires the resulting geometry to satisfy various other conditions, the possible perturbations often become quite limited. An interesting case of this is the following

Theorem. *Let g be a smooth Riemannian metric on $\mathbf{R}^2$ which differs from the canonical flat metric g_0 at most on a compact set. If $(\mathbf{R}^2, g)$ has no conjugate points, then it is isometric to $(\mathbf{R}^2, g_0)$.*

In physical terms, one may think of the Riemannian metric g as a very general type of lens, made of optically anisotropic material. A pair of conjugate points occurs when an appropriately positioned point source of light emits rays which converge at the second point. The theorem states that this refocusing of light must occur for any nontrivial lens.

R. Michel has indicated a proof of a similar result with the additional hypothesis that every geodesic of $(\mathbf{R}^2, g)$ coincides, outside a compact set, with a straight line [4]. In our case, although it is clear that positive and negative subrays of a geodesic are parts of straight lines, we do not assume that these lines are the same, or even that they are parallel. Nonetheless, our proof resembles Michel's in the strategy of showing that E. Hopf's flat torus theorem can be applied (compare Michel [5, §3.3]). Hopf showed that any Riemannian metric without conjugate points on the two-dimensional torus has vanishing Gauss curvature [3]. In a private communication to one of the authors, Michel has suggested an approach along the lines of §6 of [4] which uses Hopf's calculations but avoids appealing to the full torus theorem.

The hypotheses of the theorem are to be interpreted in the following way: If g_0 is the canonical Euclidean metric on $\mathbf{R}^2$, then we assume that $g - g_0$ has compact support. We do not assert, and it is not true, that a Riemannian metric without conjugate points on $\mathbf{R}^2$ which is *locally* Euclidean outside a compact set must be flat. For an example, one may smooth out the vertex of a

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