J. DIFFERENTIAL GEOMETRY
21 (1985) 283–300

AN INFINITE SET OF EXOTIC $\mathbf{R}^4$'S

ROBERT E. GOMPF

Introduction

In 1982, Michael Freedman startled the topological community by pointing out the existence of an "exotic $\mathbf{R}^4$", a smooth manifold homeomorphic to $\mathbf{R}^4$, but not diffeomorphic to it. This result follows easily from Donaldson's Theorem [2] on the nonexistence of certain smooth 4-manifolds, together with Freedman's powerful techniques [3] for analyzing 4-manifolds in the topological category. This exotic $\mathbf{R}^4$ was shocking to topologists, because in dimensions $n \neq 4$, it is a fundamental result of smoothing theory that there are no exotic $\mathbf{R}^n$'s. (Since $\mathbf{R}^n$ is contractible, there is no place for any bundle-theoretic obstruction to live.) Thus, this exotic $\mathbf{R}^4$ implies a catastrophic failure in dimension 4 of the basic philosophy of smoothing theory, as well as other high-dimensional techniques.

Freedman's discovery naturally raised questions about the set $\mathscr{R}$ of all oriented diffeomorphism types homeomorphic to $\mathbf{R}^4$. The most basic problem has been to determine the cardinality of $\mathscr{R}$. Soon after Freedman's result, the author showed [5] that $\mathscr{R}$ has at least four elements. More recently, Freedman and Taylor [4] have found a fifth element, a "universal" $\mathbf{R}^4$ in which all others must embed. In the present paper, we exploit a technique of Freedman and Taylor to prove that $\mathscr{R}$ is (at least countably) infinite.

Our main result, Theorem 2.3, asserts the existence of a doubly indexed family $\{R_{m,n} | m, n = 0, 1, 2, \cdots, \infty\}$ in $\mathscr{R}$ such that $R_{m,n}$ has an orientation-preserving embedding in $R_{m',n'}$ if and only if $m \leqslant m'$ and $n \leqslant n'$. In particular no two members of this family are related by an orientation-preserving diffeomorphism. We actually show that when $m > m'$ or $n > n'$ there is a compact subset of $R_{m,n}$ which cannot embed in $R_{m',n'}$.

We use these compact subsets to prove the required nonembedding property which distinguishes the $R_{m,n}$'s. Our key Lemma 1.2 gives a method for finding such compact sets which do not embed in a preassigned $R \in \mathscr{R}$. This is where

Received May 1, 1985. Supported by a National Science Foundation postdoctoral fellowship.