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THE LOCAL ISOMETRIC EMBEDDING IN R^3 OF 2-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH NONNEGATIVE CURVATURE

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0. Introduction

In this paper, we will study the local isometric embedding into R^3 of 2-dimensional Riemannian manifold. Suppose that the first fundamental form $E\,du^2 + 2F\,du\,dv + G\,dv^2$ is given in a neighborhood of p. We want to find three functions $x(u,v)$, $y(u,v)$, $z(u,v)$, such that

$$dx^2 + dy^2 + dz^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \tag{0.1}$$

in a neighborhood of p.

This embedding problem has already been solved when the Gaussian curvature K does not vanish at p. It is still an open problem when K vanishes at p. Actually, A. V. Pogorelov gave a counterexample that there exists a $C^{2,1}$ metric with no C^2 isometric embedding in R^3. In Pogorelov's example, in any neighborhood of p, there is a sequence of disjoint balls in which the metric is flat. And the Gaussian curvature K of this metric is nonnegative. The main theorem of the paper is the following.

Main Theorem. *Suppose that the Gaussian curvature of a C^s metric is nonnegative for $s \geqslant 10$, then there exists a C^{s-6} isometric embedding in R^3.*

Instead of studying the nonlinear system (0.1) of first order, we will study a second-order Monge-Ampére equation satisfied by a coordinate, say z. The equation can be derived as follows: If the Gaussian curvature of $E\,du^2 + 2F\,du\,dv + G\,dv^2 - dz^2$ vanishes, then z must satisfy

$$\begin{aligned}(z_{11} - \Gamma^i_{11}z_i)(z_{22} - \Gamma^i_{22}z_i) - (z_{12} - \Gamma^i_{12}z_i)^2 \\ = K\{EG - F^2 - Ez_2^2 - Gz_1^2 + 2Fz_1 \cdot z_2\} \equiv K(u,v,\nabla z),\end{aligned} \tag{0.2}$$

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