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EXAMPLES OF COMPLETE MANIFOLDS WITH POSITIVE RICCI CURVATURE

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Dedicated to Wilhelm Klingenberg on his sixtieth birthday

A long standing question in riemannian geometry has been: Does a complete manifold M^n with positive Ricci curvature Ric also admit a complete metric with nonnegative sectional curvature K? It is generally believed that this is not always true, but counterexamples were not known. The answer is actually affirmative for the dimension $n = 3$ (cf. [6], [16]). Note that $K > 0$ is sometimes known to be obstructed when a metric with Ric > 0 exists. Simple examples are $S^k \times \mathbf{R}^l$ in the noncompact case [5], and $\mathbf{R}P^k \times \mathbf{R}P^l$ in the nonsimply connected compact case for $k, l \geqslant 2$, as a consequence of Synge's Lemma [4].

Examples of complete manifolds with $K \geqslant 0$ remain fairly scarce. One way or another, they can all be obtained using classical spaces and quotients of isometric group actions (cf. [3] for a detailed list of references). There are several additional methods to produce complete metrics with Ric > 0. Certain fiber bundles were treated in [14] and [15], and a large class of Brieskorn varieties in [7]. Finally, by Yau's work, Kaehler metrics with Ric $\geqslant 0$ exist on any compact Kaehler manifold with first Chern class $c_1 \geqslant 0$ (cf. [17]). Interesting examples arise as complete intersections in $\mathbf{C}P^{n+r}$, notably hypersurfaces. In particular, the $K3$-surface (quartic) in $\mathbf{C}P^3$ admits a Ricci flat metric, but this is a true border line case: Since the $\hat{A}$-genus does not vanish, we have Ric $\equiv 0$ whenever Ric $\geqslant 0$ (cf. [8]). It follows that $K \geqslant 0$ would imply $K \equiv 0$, which is impossible. Therefore one can distinguish at least between the conditions Ric $\geqslant 0$ and $K \geqslant 0$, in a weak sense.

In this paper we present new classes of complete manifolds with Ric > 0. First of all we construct noncompact examples many of which cannot carry metrics with $K \geqslant 0$. This settles the above question in the noncompact case.

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