J. DIFFERENTIAL GEOMETRY
21 (1985) 143–149

THE SIMPLE LOOP CONJECTURE

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1. Introduction

The main result of this paper is the proof of the so called Simple Loop Conjecture, Theorem 2.1. In §3 we prove analogous results for compact surfaces with boundary. In that setting simple arcs play the role of simple closed curves.

I wish to thank Allen Edmonds [see 2.2] for making me aware of the applicability of my work to this problem and to thank Joel Hass and Will Kazez for helpful conversations.

Notation. If $E \subset S$, then $N(E)$ denotes a tubular neighborhood of E in S, $\mathring{E}$ denotes interior of E, and $|E|$ denotes the number of components of E. See [1] or [3] for basic definitions regarding branched covers.

2. Closed surfaces

Theorem 2.1. *If f: $S \to T$ is a map of closed connected surfaces such that f_*: $\pi_1(S) \to \pi_1(T)$ is not injective, then there exists a non contractible simple closed curve $\alpha \subset S$ such that $f|\alpha$ is homotopically trivial.*

Proof. We will assume that $T \neq S^2$.

Step 1. Either there exists a noncontractible simple closed curve $\alpha \subset S$ such that $f|\alpha$ is homotopically trivial or f is homotopic to a simple branched cover (i.e., if f is a branched cover of degree d, then for every $x \in T$ $|f^{-1}(x)| \geqslant d - 1$) or $T = \mathbf{RP}^2$ and there exists a simple branched cover f': $S \to T$ such that $\ker f_* = \ker f'_*$.

Proof of Step 1. Let D be a 2-disc in T. Let $\lambda_1, \cdots, \lambda_n$ be properly embedded arcs in $T - \mathring{D}$ such that $T - (D \mathring{\cup} N) = E$ is a 2-disc where N is a product neighborhood in $T - \mathring{D}$, of $\cup\lambda_i$.

Received February 8, 1985. Partially supported by grants from the National Science Foundation.