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ISOPARAMETRIC SUBMANIFOLDS AND THEIR COXETER GROUPS

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0. Introduction

In the later 1930's Elie Cartan defined the notion of isoparametric functions on a space form N and began their study [4]–[7]. A smooth function $f: N \to R$ ($N = \mathscr{R}^{n+1}$, S^{n+1} or H^{n+1}), is isoparametric, if Δf and $|\nabla f|^2$ are functions of f. Among other things Cartan showed that the level hypersurfaces of f are parallel, and each has constant principal curvatures. And conversely, he showed that if M is a hypersurface of N with constant principal curvatures, then there is at least a *local* isoparametric function having M as a level. Cartan called such a hypersurface isoparametric. In the last ten years, many people carried forward this research [19, 25]. Finally around 1980, Münzner [18] completed the beautiful structure theory of isoparametric hypersurfaces in the spheres, and thereby reduced their classification to a (difficult!) algebraic problem. Many people subsequently made contributions to this classification problem including U. Abresch [1], D. Ferus, H. Karcher, H. F. Münzner [15], et al. While there has been considerable recent progress, it seems much remains to be done. By and large, the theory of isoparametric hypersurfaces has been a special subject by itself; however in recent years there have been applications to the theory of harmonic maps [12], and minimal submanifolds [14, 19, 23]. Recently, Eells [12] gave a definition of isoparametric maps for the purpose of constructing harmonic maps. S. Carter and A. West [3] gave a stronger definition of isoparametric maps from N^{n+m} to R^m; their purpose being to generalize Cartan's work to higher codimension. Using their definitions, they were able to show that there is a Coxeter group (i.e., a finite group generated by reflections) associated to each isoparametric map $f: N^{n+2} \to R^2$. However, they did not obtain a similar result for larger m. They were also unable to construct a *global* isoparametric map for a given isoparametric submanifold.

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