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A REMARK ON EXTREMAL KÄHLER METRICS

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Introduction

In recent years, results from partial differential equations and differential geometry have had striking applications to the study of compact Kähler manifolds. As a notable example, the works of Aubin [1] and Yau [8], [9] on the Calabi conjecture on the existence of Kähler-Einstein metrics imply that an algebraic surface, with ample canonical sheaf and with $c_1^2 = 3c_2$ a so-called $\Phi 2$ surface, is uniformized by the ball in $\mathbf{C}^2$. This union of algebraic and differential geometry is accomplished by the existence of a particularly nice metric, whose differential geometric properties accurately reflect the complex analytic structure of the manifold.

The Kähler-Einstein metrics are solutions of a certain variational problem, introduced by Calabi in [2] and [3]. Specifically, one considers the functional S which assigns to each Kähler metric on a compact complex manifold M the integral over M of the squared scalar curvature. The functional S is restricted to the metrics with a given Kähler class ω in $H^2(M, \mathbf{R})$, and a critical point for S_ω is called an extremal Kähler metric. From the Euler equation for S_ω (see Calabi [4]), one sees that metrics with constant scalar curvature, a fortiori Kähler-Einstein metrics, are extremal. On the other hand, Calabi has exhibited algebraic surfaces which have an extremal metric, but have no metric of constant scalar curvature.

The purpose of this note is to exhibit examples of compact Kähler manifolds which do not admit an extremal Kähler metric. The recent work of Calabi [5] includes a structure theorem for the group of holomorphic automorphisms of a Kähler manifold M which has an extremal metric; in particular, if the dimension of the automorphism group of M is positive, the group must contain a nontrivial compact real Lie subgroup. The examples given here all fail to have such a compact subgroup of their automorphism group. This does not

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