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ON CARNOT-CARATHÉODORY METRICS

JOHN MITCHELL

1. Introduction

Consider a smooth Riemannian n-manifold (M, g) equipped with a smooth distribution of k-planes. Such a distribution Δ assigns to each point $m \in M$ a k-dimensional subspace of the tangent space $T_m M$. An absolutely continuous curve α in M is said to be horizontal if it is a.e. tangent to the distribution Δ. One may define a metric on M as follows.

Definition. The Carnot-Carathéodory distance between two points $p, q \in M$ is $d_c(p, q) = \inf_{\omega \in C_{p,q}}\{\text{length}(\omega)\}$, where $C_{p,q}$ is the set of all horizontal curves which join p to q. The metric d_c is finite provided that the distribution Δ satisfies Hörmander's condition (assuming that M is connected). To describe this condition, let $X_1, X_2, \cdots, X_k$ be a local basis of vector fields for the distribution near $m \in M$. If these vector fields, along with all their commutators, span $T_m M$, then the vector fields are said to satisfy Hörmander's condition at m. Denote by $V_i(m)$ the subspace of $T_m M$ spanned by all commutators of the X_j's of order $\leqslant i$ (including, of course, the X_j's). It is easy to see that $V_i(m)$ does not depend upon the choice of local basis $\{X_j\}$, so it makes sense to say that the distribution satisfies Hörmander's condition at m if $\dim V_i(m) = \dim(M)$ for some i. This infinitesimal transitivity implies local transitivity:

Theorem (*Chow*). *If a smooth distribution satisfies Hörmander's condition at $m \in M$, then any point $p \in M$ which is sufficiently close to m may be joined to m by a horizontal curve.*

Thus, if M is connected, the metric d_c is finite.

We will prove below the following two local theorems concerning the metric space (M, d_c) associated to a generic distribution Δ on M. (A distribution is said to be *generic* if, for each i, $\dim(V_i(m))$ is independent of the point

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