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THE HEAT EQUATION ON A CR MANIFOLD

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0. Introduction

The trace of the heat semigroup for the Laplacian on a compact oriented Riemannian manifold has an asymptotic expansion in powers of the time t for small positive t whose coefficients are integrals of local geometric invariants (see [1], [6], [11] and their references). This expansion and its generalizations to other elliptic operators have been powerful tools in the study of the relationship between analysis and geometry on the manifold (see the surveys in [4], [12] and [16]).

In this paper we prove analogous results for the sublaplacian $\square_b$ on a compact CR manifold. The classical pseudodifferential calculus is not adequate for this purpose because $\square_b$ is not elliptic, so we develop an appropriate pseudodifferential calculus here. To motivate a description of our methods and results we begin with a sketch of a proof of the Riemannian result along similar lines (see [7] for details). We then point out the differences due to difficulties in carrying the program over to the nonelliptic case.

Let M be a compact oriented Riemannian manifold and let $\Delta = d^*d$ denote the Laplace-Beltrami operator on functions. In local coordinates

$$\Delta = -\sum_{i,j} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} g^{ij} \sqrt{g} \frac{\partial}{\partial x^j}, \tag{0.1}$$

where the metric tensor is given by the matrix (g_{ij}) which has inverse g^{ij} and $g = \det(g_{ij})$. Let $P = \partial/\partial t + \Delta$ operating on functions on $M \times \mathbf{R}$. We seek to construct a parametrix for P, i.e. a pseudodifferential operator Q such that $PQ \equiv QP \equiv I$ modulo smoothing operators. If Q is a parametrix it is given by

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