J. DIFFERENTIAL GEOMETRY
20 (1984) 311–327

THE EQUATION OF PRESCRIBED GAUSS CURVATURE WITHOUT BOUNDARY CONDITIONS

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Abstract

A necessary condition for the equation of prescribed Gauss curvature to have a convex solution defined on a domain $\Omega \subset \mathbf{R}^n$ is that the Gauss curvature K satisfies $\int_\Omega K \leqslant \omega_n$. We prove the existence, uniqueness and regularity, under suitable hypotheses, of a convex solution in the extremal case $\int_\Omega K = \omega_n$. We also discuss the boundedness of convex solutions of the equation.

1. Introduction

In this paper we are concerned with the existence, uniqueness, regularity and boundedness of convex solutions of the equation of prescribed Gauss curvature

$$\det D^2 u = K(x)\left(1 + |Du|^2\right)^{(n+2)/2} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbf{R}^n$ without imposing boundary conditions on the function u. In particular, we are interested in a certain extremal case.

Equation (1.1) is elliptic only for functions $u \in C^2(\Omega)$ which are uniformly convex at each point of Ω. For such solutions to exist we must therefore assume that K is positive in Ω.

Suppose that $u \in C^2(\Omega)$ is a uniformly convex solution of (1.1). Then the gradient mapping Du is one-to-one with Jacobian $\det D^2 u$, so by integrating (1.1) and changing variables, we obtain, as in [4],

$$\begin{aligned}\int_\Omega K &= \int_{Du(\Omega)} \frac{dp}{\left(1 + |p|^2\right)^{(n+2)/2}} \\ &\leqslant \int_{\mathbf{R}^n} \frac{dp}{\left(1 + |p|^2\right)^{(n+2)/2}} = \omega_n,\end{aligned} \tag{1.2}$$

Received December 5, 1983 and, in revised form, September 4, 1984.