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EXAMPLES OF SIMPLY-CONNECTED SYMPLECTIC NON-KÄHLERIAN MANIFOLDS

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1. Introduction

A symplectic manifold (M, σ) is a pair consisting of a $2n$-dimensional manifold M together with a closed 2-form σ which is nondegenerate (that is, σ^n never vanishes). The form σ determines two pieces of topological data: the cohomology class $a = [\sigma] \in H^2(M; \mathbf{R})$ and a homotopy class of reductions of the structural group of the tangent bundle of M to $U(n) \simeq \mathrm{Sp}(2n; \mathbf{R})$, and hence a homotopy class $[J]$ of almost complex structures on M. Gromov showed in his thesis that, if M is open, any such pair $(a, [J])$ may be realised by some symplectic form (see [3], [4]). If M is closed, a^n must be a generator of $H^{2n}(M; \mathbf{R})$ which is positive with respect to the orientation defined by $[J]$. But even with this condition, it is not known whether the corresponding statement is true. In fact, very few closed symplectic manifolds are known. Any Kähler manifold is symplectic. Thurston showed in [6] how to construct a non-Kähler closed symplectic manifold. His examples are nil-manifolds and so are not simply-connected. (A similar example was known to Serre. See [10], Problem 42. Thurston's construction is further developed in [9] and [3].)

In [3] Gromov points out that if the symplectic manifold (M, σ) is symplectically embedded in (X, ω), then one can "blow up" X along M to obtain a new symplectic manifold $(\tilde{X}, \tilde{\omega})$. In this note we use this technique together with a symplectic embedding theorem (see [5], [2], [7]) to construct some simply-connected, closed symplectic manifolds which are not Kähler.

Here is one such example. Let (M, σ) be Thurston's 4-dimensional symplectic, but non-Kähler manifold. It is the quotient $\mathbf{R}^4/\Gamma$, where Γ is the discrete affine group generated by the unit translations along the x_1, x_2, x_3-axes together with the transformation $(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_2, x_2, x_3, x_4 + 1)$. Thus M is a T^2-bundle over T^2. Its symplectic form σ lifts to $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ on $\mathbf{R}^4$. Note that σ is integral, that is, $[\sigma] \in H^2(M; \mathbf{Z})$. Further,

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