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FINITE VOLUME AND FUNDAMENTAL GROUP ON MANIFOLDS OF NEGATIVE CURVATURE

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1. Introduction

Let V be a complete Riemannian manifold of dimension n and sectional curvature $K \leqslant 0$. Then V is a $K(\pi, 1)$-manifold with $\pi = \pi_1(V)$ [8, p. 103] and hence determined up to homotopy by the fundamental group. In particular, the homology $H_*(V)$ of V is isomorphic to the group homology $H_*(\pi_1(V))$ (see [1]). Therefore V is compact if and only if $H_n(\pi_1(V), \mathbf{Z}_2) = \mathbf{Z}_2$. Hence the compactness of V can be read off from $\pi_1(V)$.

We give a similar characterization for the condition of finite volume:

Theorem. *Let V be a complete Riemannian manifold of dimension $n \geqslant 3$ with curvature $-b^2 \leqslant K \leqslant -a^2 < 0$. Then the volume of V is finite if and only if*:

(1) *$\pi_1(V)$ contains only finitely many conjugation classes of maximal almost nilpotent subgroups of rank $n - 1$.*

(2) *If Δ is the amalgamated product of $\pi_1(V)$ with itself on these subgroups, then $H_n(\Delta, \mathbf{Z}_2) = \mathbf{Z}_2$.*

For a full definition of Δ we refer to §4.

For $n = 2$, the statement is wrong: Let V be a noncompact surface with constant negative curvature and finite volume. It is known that V has an end E diffeomorphic to $S^1 \times (0, \infty)$ with a warped product metric $f^2\, ds^2 + dt^2$. The curvature is given by $-f''/f$ and the volume of E by $2\pi\int_0^\infty f\, dt$. Using a suitable function $\bar{f}$ we can deform E to an expanding end, such that the new end has bounded negative curvature but infinite volume.

The first part of our proof (§3) leads to a description of the ends of finite volume in terms of the fundamental group. This part is based on the investigations of Heintze [6], Gromov [5] and Eberlein [3]. A topological argument then finishes the proof (§4).

This paper is a condensed version of parts of my thesis [10] written under the guidance of Professor Wolfgang Meyer at Münster. I am also deeply

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