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# TOPOLOGICAL INVARIANTS AND EQUIDESINGULARIZATION FOR HOLOMORPHIC VECTOR FIELDS

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The integrals of a holomorphic vector field $Z$ defined in an open subset $\mathcal{U}$ of $\mathbf{C}^n$ are complex curves parametrized locally as the solutions of the differential equation

$$\frac{dz}{dT} = Z(z), \qquad T \in \mathbf{C}, z \in \mathcal{U}.$$

They define a complex one-dimensional foliation $\mathcal{F}_Z$ of $\mathcal{U}$ with singularities at the zeros of $Z$. The purpose of this paper is to exhibit several topological invariants of these foliations near a singular point.

Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions defined in some neighborhood of $0 \in \mathbf{C}^n$ and let $I(Z_1,\cdots,Z_n)$ be the ideal generated by the germs at $0 \in \mathbf{C}^n$ of the coordinate functions of $Z$. We define the *Milnor number* $\mu$ of the vector field $Z$ at $0 \in \mathbf{C}^n$ as

$$\mu = \dim_{\mathbf{C}} \mathcal{O}_n / I(Z_1,\cdots,Z_n).$$

This number is finite if and only if $0 \in \mathbf{C}^n$ is an isolated singularity of $Z$, a hypothesis which we will assume from now on. In this case $\mu$ coincides with the topological degree of the Gauss mapping induced by $Z$, considered as a real vector field, in a small $(2n-1)$-sphere around $0 \in \mathbf{C}^n$. In Theorem A we show that: *the Milnor number of $Z$ is a topological invariant of $\mathcal{F}_Z$ provided that $n \geqslant 2$.*

Consider now a polydisc $B \subset \mathcal{U}$ centered at $0 \in \mathbf{C}^n$ and let $f\colon B \to \mathbf{C}^k$, $f(0) = 0$, be an irreducible analytic function. Then $V = f^{-1}(0)$ is an analytic subvariety and we say *$V$ is invariant by $Z$ if for any $p \in V$ we have $df(p) \cdot Z(p) = 0$.* Suppose $k = n-1$. Then $\dim_{\mathbf{C}} V = 1$ and $V - \{0\}$ is a leaf of $\mathcal{F}_Z$. Moreover, if $B$ is small enough, then $D = B \cap V$ is homeomorphic to a 2-disc via a Puiseaux's parametrization. Then the restriction of $Z$ to $D$ can be considered as a real vector field $X$ defined in a 2-disc. *The multiplicity of $Z$*

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