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A DUALITY THEOREM FOR WILLMORE SURFACES

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0. Introduction

In 1965 T. J. Willmore [12] proposed to study the functional

$$\tilde{\mathscr{W}}(X) = \int_M H^2 \, dA$$

on immersions X: $M^2 \to \mathbf{E}^3$, where M^2 is a compact surface, H is the mean curvature of the immersion, and dA is the induced area from (or area density if M is not oriented). If we define

$$\mathscr{W}(X) = \int_M (H^2 - K) \, dA,$$

then by the Gauss-Bonnet theorem

$$\tilde{\mathscr{W}}(X) = \mathscr{W}(X) + 2\pi\chi(M),$$

so the two functionals differ by a constant. The functional $\mathscr{W}(X)$ has the advantage that its integrand is nonnegative and vanishes exactly at the umbilic points of the immersion X.

Obviously $\mathscr{W}(X) = 0$ iff $M^2 = S^2$ and X is totally umbilic. Thus, the absolute minimum of $\mathscr{W}$ on the space of immersions X: $S^2 \to \mathbf{E}^3$ is 0 and the critical locus of such X is known. When M is a torus, Willmore provided an example of an immersion X: $M \to \mathbf{E}^3$ with $\mathscr{W}(X) = 2\pi^2$ and showed that $\mathscr{W}(X) \geqslant 2\pi^2$ for all smooth surfaces of revolution. He then conjectured that $\mathscr{W}(X) \geqslant 2\pi^2$ for all immersions of the torus with equality only for the example he provided: the anchor ring swept out by revolving a circle of radius r about the line whose distance from the center of the circle was $r\sqrt{2}$. White then pointed out that the two-form $(H^2 - K)\, dA$ had the property of being invariant under conformal transformations of $\mathbf{E}^3$ plus the "point at infinity"

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