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CLOSED GEODESICS ON MANIFOLDS WITH INFINITE ABELIAN FUNDAMENTAL GROUP

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While many results are known about closed geodesics on compact manifolds with finite fundamental group, there are few for manifolds with π_1 infinite ([2], [6]), with the exception of manifolds admitting metrics of nonpositive curvature. We prove that for every compact Riemannian manifold M with fundamental group $\pi_1 = \mathbf{Z}$ the number of geometrically distinct closed geodesics of length $\leqslant l$ grows at least like the prime numbers. In particular, there are infinitely many. Geodesics are called geometrically distinct if their images on M are different.

Manifolds with $\pi_1 = \mathbf{Z}$ can be found, e.g., by taking a bundle over S^1 with simply connected fiber or by attaching a handle $I \times S^{n-1}$ to a simply connected manifold of dimension $n \geqslant 3$. There are no examples in dimension 2.

Every nontrivial free homotopy class of closed curves on a compact manifold contains a curve of minimal length which is a closed geodesic. If $\pi_1 = \mathbf{Z}$ there are infinitely many free homotopy classes but the corresponding minimal length geodesics can be geometrically indistinct. If $\pi_1(M)$ is abelian and Rank $\pi_1 \otimes \mathbf{Q} \geqslant 2$, then the fundamental group is enough to ensure the existence of infinitely many closed geodesics: if $s, t \in \pi_1$ are independent and of infinite order, minimal length curves in the classes st^m will be geometrically distinct. While those in the classes t^m may not be distinct, the following generalization of our theorem is true:

Suppose $\pi_1(M)$ is abelian and $t \in \pi_1$ has infinite order. Then the number of geometrically distinct closed geodesics of length $\leqslant l$ in the classes t^m, $m \geqslant 1$, grows at least like the prime numbers.

The case where π_1 is infinite abelian but $\pi_1 \neq \mathbf{Z}$ is easier and is already known to some experts. We will include a proof for this case at the end of the paper.

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