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EQUIVARIANT MORSE THEORY AND CLOSED GEODESICS

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In this paper we use the equivariant Morse theory to investigate the existence of closed geodesics on simply connected compact Riemannian manifolds, particularly the simply connected compact rank 1 symmetric spaces (CROSS's) S^n, $\mathbf{C}P^n$, $\mathbf{H}P^n$, CaP^2.

In §5 we show the existence of at least $g(\lambda, n)$ "short" closed geodesics without self-intersection on a homotopy CROSS sufficiently close to the standard metric, where $g(\lambda, n)$ is the cuplength of the space of unparameterized geodesics on the standard CROSS. In the nondegenerate case there will be $\lambda(\lambda + 1)n(n + 1)/4$ $(\lambda = 1, 2, 4, 8)$.

In §6 we prove that if M is a simply connected compact Riemannian manifold with the rational homotopy type of a CROSS and if all closed geodesics on M are hyperbolic, then the number of distinct closed geodesics of length $\leqslant l$ on M grows at least like the prime numbers.

Birkhoff gave a proof of the existence of at least one closed geodesic on an n-sphere in 1927. In 1929 Lusternik and Schnirelmann claimed the existence of three closed geodesics without self-intersection for any metric on a 2-sphere. The proof however was only completed recently by Ballmann. The geodesics of these theorems are obtained by shortening deformations of the great and small "circles" on a sphere; such geodesics can be considered "short". The generalization to higher dimensional spheres was attempted by Alber, who gave a proof of the existence of $g(n)$ closed geodesics on an n-sphere, where $g(n)$ is the cuplength of the Grassmannian $G(2, n + 1)$. But Ballmann found a mistake in Alber's proof. Ballmann, Thorbergsson and Ziller ([11], [12]), Anosov [2] and the author [28] independently gave complete proofs of versions of Alber's theorem. In the nondegenerate case one obtains $n((n + 1)/2)$ closed geodesics on a sphere. Morse [37] showed that the latter number is optimal with the example of an ellipsoid with unequal axes. While such an ellipsoid will

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