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MULTIPLICITY-FREE SPACES

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Introduction

Let G be a Lie group. A unitary representation of G on a Hilbert space, H, is called multiplicty-free if every irreducible representation of G occurs in H with multiplicity zero or one. It is easy to see that this is the case if and only if the ring of bounded G-invariant operators on H is commutative. In this paper we will examine the symplectic analogue of this situation: Let X be a symplectic manifold on which G acts in a Hamiltonian fashion. If one thinks of the bounded operators on H as "quantum observables" and the functions on X as "classical observables" the analogue of the situation above is that the ring of G-invariant functions on X be commutative with respect to Poisson-bracket. If this happens we will say that X is *multiplicity-free*. We were led to the study of such manifolds by some questions in dynamical systems. Let $\Phi : X \to \mathfrak{g}^*$ be the moment mapping. A function of the type $f \circ \Phi$, for $f\colon \mathfrak{g}^* \to \mathbf{R}$, is called *collective* (cf. [6]); a completely integrable system consisting of functions of this type is called a *collective* completely integrable system (see [12] or [20]). We noticed [12], that a necessary condition for X to admit a collective completely integrable system is that it be multiplicity-free. We also proved that for certain groups, in particular for $U(n)$ and $O(n)$, this condition is sufficient as well.

This paper will consist of two parts. In part one we will study the local structure of multiplicity-free spaces in the neighborhood of a fixed coisotropic orbit. We will confine ourselves to the case where G is compact and connected, though many of our results are true more generally. Our first main result will be that the problem of determining the local structure of such spaces, up to isomorphism, can be reduced to the special case where X is a cotangent bundle. This reduction will involve two pieces of symplectic engineering which are of considerable interest in their own right. The first, the "induction" construction, is well known, but we will present it here in a somewhat unfamiliar guise. The second, the "cross-section" construction, was partly inspired by the material in

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