J. DIFFERENTIAL GEOMETRY
18 (1983) 701–721

THE DIRICHLET PROBLEM AT INFINITY FOR MANIFOLDS OF NEGATIVE CURVATURE

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This paper is concerned with the existence of bounded harmonic functions on simply connected manifolds N^n of negative curvature. It has been conjectured for some time with such manifolds admit a wealth of bounded harmonic functions provided the sectional curvature K_N satisfies $-a^2 \leqslant K_N \leqslant -b^2$, for some constants, a, $b > 0$, or even if $K_N \leqslant -b^2 < 0$; see [7], [18]. Justification for this comes from the fact that the model space $H^n(-1)$, the space form of curvature -1, admits many bounded harmonic functions; in fact, there is a Poisson integral representation 'at infinity' in $H^n(-1)$. (Similar results hold in more general noncompact symmetric spaces, cf. [12].) Furthermore, in case $n = 2$ the Ahlfors-Schwarz Lemma [1] shows that N^2 is conformally the unit disc provided $K_N \leqslant -b^2 < 0$, so that the model $H^2(-1)$ provides full information in this case.

It is natural to consider a Dirichlet problem at infinity for the Laplace-Beltrami operator Δ on N^n; there is a well-known compactification $\overline{N^n} = N^n \cup S^{n-1}(\infty)$ of N^n giving a homeomorphism of $(N^n, S^{n-1}(\infty))$ with the Euclidean pair (B^n, S^{n-1}). One can then state the

Asymptotic Dirichlet problem for Δ. Given a continuous function ρ on $S^{n-1}(\infty)$, find $f \in C^\infty(N^n) \cup C^0(\overline{N^n})$ satisfying

$$\Delta f = 0, \qquad f|_{S^{n-1}(\infty)} = \rho.$$

The main result of this paper is given by the following theorem (Theorem 3.2).

Theorem. *Let N^n be a complete simply connected Riemannian manifold with sectional curvature K_N satisfying $-a^2 \leqslant K_N \leqslant -b^2$, where $a^2 \geqslant b^2$ are arbitrary positive constants. Then the asymptotic Dirichlet problem for Δ is uniquely solvable, for any $\rho \in C^0(S^{n-1}(\infty))$.*

In particular, it follows that N^n has a large class of bounded harmonic functions. Using this one may show for instance that there are smooth proper

Received November 11, 1982 and, in revised forms, December 6, 1982, March 17 and April 15, 1983. Research supported in part by NSF Grant MCS 81-04235.