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OSCULATION BY ALGEBRAIC HYPERSURFACES

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Introduction

In this paper we give necessary and sufficient conditions for d pieces of hypersurface to be osculated to a fixed order by an algebraic hypersurface of degree d.

Given a line L_0 in P^{n+1} and d points $P_1^0,\cdots,P_d^0$ on L_0, suppose there are d pieces of hypersurface $\gamma_1,\cdots,\gamma_d$ such that $P_i^0 \in \gamma_i$ and L_0 intersects each γ_i transversely.

The question addressed here is: when does there exist an algebraic hypersurface γ of degree d which osculates each piece γ_i to order r at P_i^0? The main result gives necessary and sufficient conditions for the existence of such an algebraic hypersurface γ.

Fix affine coordinates $(x_0,\cdots,x_n)$ on P^{n+1}, and fix line coordinates $(m_1,\cdots,m_n, b_1,\cdots,b_n)$, where a line L is given by $x_k = m_k x_0 + b_k$, $k = 1,2,\cdots,n$. (Line coordinates are just local coordinates on $\mathrm{Gr}(1, n+1)$, the Grassmannian of all lines in P^{n+1}.) Assume that coordinates have been chosen so that the given line L_0 has line coordinates $m_k = 0$, $b_k = 0$ for all k. L_0 is then the x_0-axis. For convenience, write $m = (m_1,\cdots,m_n)$, $b = (b_1,\cdots,b_n)$. A line $L = L(m, b)$ near L_0 will intersect each γ_i at a point $P_i = P_i(m, b)$. Then $P_i(0,0) = P_i^0$. Let $X_i = X_i(m, b)$ be the 0th coordinate of P_i in terms of the affine coordinate system. Define $K_{jk} = K_{jk}(m, b)$ by

$$K_{jk}(m, b) = \frac{\partial^2[\Sigma_i X_i(m, b)]}{\partial b_j \partial b_k}$$

$j, k = 1,2,\cdots,n$. We can now state the main result.

Theorem. *There exists an algebraic hypersurface γ of degree d, which osculates each γ_i to order r, $2 \leqslant r \leqslant d$, at P_i^0, $i = 1,2,\cdots,d$, if and only if K_{jk} and all of its partial derivatives of order $\leqslant r-2$ vanish at $(m, b) = (0,0)$.*

Remarks. 1. If the order of osculation desired is $r = 0$ or $r = 1$, there is no condition. Just take γ to be the union of the d tangent hyperplanes to γ_i at P_i^0.

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