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THE FILLING RADIUS OF TWO-POINT HOMOGENEOUS SPACES

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Let X be a closed connected manifold of dimension n, and let dist $=$ dist(x, x') be a Riemannian metric on X. The function d_x on X given by $d_x(x') = \operatorname{dist}(x, x')$ lies in the space $L^\infty(X)$ of all bounded functions on X with the sup-norm $\|\ \|$. The canonical inclusion

$$X \to L^\infty(X), \quad x \mapsto d_x$$

is an isometric imbedding, as $\operatorname{dist}(x, x') = \|d_x - d_{x'}\|$ (the triangle inequality). Consider the inclusion homomorphism $\alpha_\varepsilon \colon H_n(X) \to H_n(U_\varepsilon X)$, where $U_\varepsilon X \subset L^\infty(X)$ is the ε-neighborhood of X, and the coefficients are in $\mathbf{Z}_2$. Following M. Gromov [3], we introduce a new metric invariant of X.

Definition. The *filling radius* of X, denoted Fill Rad X, is the infimum of those $\varepsilon > 0$ for which $\alpha_\varepsilon([X]) = 0$, where $[X]$ is the fundamental class of X.

We prove the following theorems.

Theorem 1. *The filling radius of the real projective space* $\mathbf{R}P^n$ *of constant curvature* $+1$ *equals one third of its diameter*:

$$\operatorname{Fill\,Rad} \mathbf{R}P^n = \frac{1}{3} \operatorname{diam} \mathbf{R}P^n = \frac{\pi}{6}.$$

Theorem 2. *The filling radius of the sphere* S^n *of constant curvature* $+1$ *equals one half of the spherical distance between two vertices of an inscribed regular* $(n+1)$*-simplex*:

$$\operatorname{Fill\,Rad} S^n = \frac{1}{2} \arccos\left(-\frac{1}{n+1}\right).$$

We also obtain partial results for the projective spaces over the complex numbers, the quaternions, and the Cayley numbers (see Propositions 1–3). Our estimates from below for the filling radius of two-point homogeneous spaces depend on a version of Jung's theorem (see Lemma 2) and the Alexandrov-Toponogov comparison theorem (see Lemma 3).

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