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THREE EXOTIC R^4'S AND OTHER ANOMALIES

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Abstract

We show that there are at least four smooth structures on $\mathbf{R}^4$, and that there are at least four oriented diffeomorphism types of Casson handles. This is also an expository paper concerning the theorems of Freedman and Donaldson; these theorems interact to demonstrate anomalous behavior for 4-manifolds.

0. Introduction

There have recently been two quantum jumps in our understanding of 4-manifolds. Mike Freedman's work [4] has shown that topological 4-manifolds behave much like higher dimensional manifolds, at least in the simply connected case (with hope for the nonsimply connected case as well). In sharp contrast, Simon Donaldson's theorem [2] shows that smooth 4-manifolds behave in a radically different way. In particular, Freedman's main results, surgery and h-cobordism theorems, have counter-examples in the smooth category. The most dramatic example of this pathology is the existence of exotic smooth structures on $\mathbf{R}^4$, in particular, smooth manifolds which are homeomorphic but not diffeomorphic to $\mathbf{R}^4$.

These exotic $\mathbf{R}^4$'s are surprising for several reasons. First, it is a standard fact that for $n \neq 4$, exotic $\mathbf{R}^n$'s cannot exist. More fundamentally, exotic $\mathbf{R}^4$'s show the failure in the smooth category of several major theories of higher dimensional manifolds. We see the breakdown of surgery theory (even in the simply connected case) during the construction of an exotic $\mathbf{R}^4$ (see §1). There can be no 5-dimensional proper h-cobordism theorem, as it is easy to construct a smooth h-cobordism from any exotic $\mathbf{R}^4$ to the standard one. (This cannot be a smooth product $\mathbf{R}^4 \times I$, as one boundary component is not diffeomorphic to $\mathbf{R}^4$.) Finally, we see a major breakdown of higher dimensional smoothing theory, as we will now explain.

In dimensions five and up, the number of smooth structures on a given manifold is determined by certain cohomology groups [6]. For example,

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