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A NEW PROOF OF A THEOREM OF NARASIMHAN AND SESHADRI

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1. Introduction

In 1965 Narasimhan and Seshardri proved that the stable holomorphic vector bundles over a compact Riemann surface are precisely those arising from irreducible projective unitary representations of the fundamental group [5]. We shall give here a different, more direct, proof of this fact using the differential geometry of connections on holomorphic bundles. This complements, in a small way, the recent paper by Atiyah and Bott [1] in which the result of Narasimhan and Seshadri is used to calculate the cohomology of the moduli spaces of stable bundles, and which we take as a general reference for background and notation.

Let X be a compact Riemann surface with a Hermitian metric normalized to unit volume. If E is a vector bundle over X we write

$$\mu(E) = \text{degree(E)}/\text{rank}(E),$$

where the degree is obtained by evaluating $c_1(E)$ on the fundamental cycle. A holomorphic bundle $\mathcal{E}$ is defined to be *indecomposable* if it cannot be written as a proper direct sum, and to be *stable* if for all proper holomorphic sub-bundles $\mathcal{F} < \mathcal{E}$,

$$\mu(\mathcal{F}) < \mu(\mathcal{E}), \text{ or equivalently } \mu(\mathcal{E}/\mathcal{F}) > \mu(\mathcal{E}).$$

Certainly a stable bundle is indecomposable. The theorem to be proved is:

Theorem. *An indecomposable holomorphic bundle $\mathcal{E}$ over X is stable if and only if there is a unitary connection on $\mathcal{E}$ having constant central curvature $*F = -2\pi i\mu(\mathcal{E})$. Such a connection is unique up to isomorphism.*

Note. If $\deg(\mathcal{E}) = 0$, these connections are flat and so are given by unitary representations of the fundamental group. In the general case it is easy to prove the equivalence of this form of the result with the statement of Narasimhan and Seshadri [1, §6].

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