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POINCARÉ'S PROBLEM AND THE LENGTH OF THE SHORTEST CLOSED GEODESIC ON A CONVEX HYPERSURFACE

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0. Introduction

In this paper we discuss two different but related problems.

The first problem is that of finding upper and lower bounds on the length L of the shortest nontrivial closed geodesic on a convex hypersurface $M^n \subset \mathbf{R}^{n+1}$. We show that if M encloses a ball of radius r_0, then $L \geqslant 2\pi r_0$ (Theorem 1.5). We also show (Theorem 1.7) that

$$L \leqslant \frac{2\pi}{\sqrt{n}\,\alpha(n)} \int_M \sqrt{S_{n-1}\big(a_1^2(x), \cdots, a_n^2(x)\big)}\, dx,$$

where $\alpha(n)$ is the volume of the unit n sphere, $\{a_i(x)\}$ is the set of principal curvatures of M at x, and S_{n-1} is the $(n-1)$st symmetric polynomial. Further, if equality holds (in the upper bound), then M is a round sphere. The upper bound is interesting in that it is in terms of an integral of curvatures rather than bounds on curvatures. The lower bound is used in the proof of the second problem.

The second problem was posed by H. Poincaré, in 1905, in a well-known paper [6]. In [6] it was suggested that one could find the shortest simple closed geodesic on a convex surface M by minimizing the arclength functional over the set $\mathcal{C}$ of all simple smooth closed curves which separate M into two pieces of equal total curvature. Here we establish that this suggestion in fact works.

In 1980 using the methods of integral currents M. S. Berger and E. Bombieri [1] showed that the result holds for metrics C^3 close to the standard metric. The reason for this restriction comes in showing that the minimum which they get is connected. They suggest that by complicating the proofs and using the theory of varifolds one may be able to extend their proof to cover all convex surfaces.

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