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ENDS OF MAPS. III: DIMENSIONS 4 AND 5

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Topology in dimension 4 has recently been considerably illuminated by Freedman's embedding theorem for topological 2-handles, and even more recently by Donaldson's nonexistence theorem for smooth structures. This paper complements Donaldson's work, and extends Freedman's to the topological category, by proving (partial) existence and uniqueness theorems for smooth structures. The main theorems are actually 5-dimensional versions of the thin h-cobordism and end theorems proved in dimensions $\geqslant 6$ in Ends of Maps, I [13]. These theorems are stated in §2.1, and proved in §3. They directly imply a number of useful facts about 4- and 5-manifolds, which we summarize:

In §2.2 we see that a homeomorphism of smooth 4-manifolds is isotopic to one which is smooth off a "standard singular set". Applying this to the special case of an open handle shows 0 and 1 handles can be straightened, and 2-handles can be straightened in several weak senses. (The 0-handle case is the classical "annulus conjecture".) This implies (via immersion theory) that the stability map $\mathrm{TOP}(4)/O(4) \to \mathrm{TOP}/O$ is 3-connected. In turn this implies that every 4-manifold has a smooth structure in the complement of a point, extending the canonical one on the boundary. In particular "almost smooth" can be deleted from the statement of Freedman's classification theorem [6, Theorem 1.5].

§2.3 demonstrates that topological 5-manifolds have handlebody structures, relative to arbitrary submanifolds of their boundary. This completes this problem: all topological manifold pairs $(M, \partial_0 M)$ have handlebody structures except nonsmoothable 4-manifolds.

In §2.4 most of the (few) remaining cases of topological transversality are settled. Map transversality holds without exception, but two cases of isotopy transversality of submanifolds remain undecided.

In §2.5 the homotopy characterization of local flatness is extended to dimension 4, except in codimension 2.

In §2.6 some of the general theory of cell-like maps is extended to dimension 4. The fact that a cell-like map of manifolds can be approximated by

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