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CONFORMAL AND MINIMAL IMMERSIONS OF COMPACT SURFACES INTO THE 4-SPHERE

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ABSTRACT. We study the twistor map of Penrose, T: $\mathbf{CP}^3 \to S^4$ and show that the complex 2-plane field in $\mathbf{CP}^3$ orthogonal to the fibers of T is a holomorphic nonintegrable 2-plane field. We then show that every horizontal holomorphic curve in $\mathbf{CP}^3$ projects under T to be a minimal surface in S^4. Finally, we use the Riemann-Roch theorem to construct, for any compact Riemann surface M^2, a holomorphic horizontal curve Φ: $M^2 \to \mathbf{CP}^3$ without ramification. It follows that $T \circ \Phi$: $M^2 \to S^4$ is a conformal and minimal immersion.

0. Introduction

The study of minimal surfaces in spheres has received much attention. In [7] Lawson proved that every compact surface except $\mathbf{RP}^2$ could be immersed into S^3 as a minimal surface. However, it is unknown whether every compact *Riemann* surface (= compact surface with a fixed complex structure) can be conformally and minimally immersed into S^3.

In [2], [3] Calabi studied minimal surfaces in Euclidean spheres, and in [4], [5] Chern studied minimal immersions of the two-sphere into S^4 and more general spaces of constant curvature. Given a minimal immersion X: $M^2 \to S^n$, where M^2 is assumed oriented and is given the unique complex structure compatible with the orientation and the metric on M^2 induced by the immersion, they found that they could construct a holomorphic quartic form Q_X on M^2 from the second fundamental form of the immersion. If $M^2 = S^2$, then the Riemann-Roch theorem shows that $Q_X \equiv 0$. Exploiting this fact, Calabi and Chern were able to prove extensive results concerning minimal immersions of S^2 into S^n. In the present paper, immersions X: $M^2 \to S^n$ satisfying $Q_X \equiv 0$ are referred to as *superminimal* immersions. In an unpublished work, the author has shown that the over-determined system of partial differential equations whose solutions are the superminimal immersions is *involutive* in Cartan's sense, so one expects a good local theory.

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