J. DIFFERENTIAL GEOMETRY
17 (1982) 233–238

A LOWER BOUND FOR THE FIRST EIGENVALUE OF A NEGATIVELY CURVED MANIFOLD

RICHARD SCHOEN

There has been much work in recent years on the relation of the low eigenvalues of a compact Riemannian manifold to the geometry of the manifold. For Riemann surfaces with positive genus, it was observed by P. Buser [1] that one can find a compact hyperbolic surface of fixed genus (hence fixed area) with arbitrarily small first eigenvalue (see [10] for more information on this problem). For hyperbolic manifolds of dimension larger than two, Mostow's theorem implies that the topology uniquely determines the geometry, so the above phenomenon for λ_1 is likely to be a two-dimensional phenomenon. In this note we show that this is the case. Precisely, let M^n be a compact Riemannian manifold with sectional curvature bounded between two negative constants. We show here that if $n \geqslant 3$, then $\lambda_1(M)$ has a lower bound depending only on the volume of M. Actually, for $n > 3$, Gromov [7] has shown that an upper bound on volume implies an upper bound on diameter (for negatively curved M). Using this result, a bound such as ours would follow from a general result of S. T. Yau [11]. For $n = 3$, the diameter is not bounded in terms of volume (see [2, 3.13]) so our result seems to be of most interest in this case. Buser [2] has observed that our dependence on the inverse square of the volume is best possible.

The case $n = 3$ of our theorem was announced in the Hawaii Symposium in 1979. In this note we give a simplified version, valid for all $n > 2$, of our original proof. We wish to thank P. Buser for pointing out reference [9] which is used in the proof of Lemma 1.

The main results

We will assume throughout that M^n is a compact n dimensional manifold. We state our main result.

Received December 14, 1981. Research supported by the Sloan Foundation.