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SUBMANIFOLDS AND SPECIAL STRUCTURES ON THE OCTONIANS

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0. Introduction

Geometries associated to the exceptional groups and "exceptional" representations of classical groups often display interesting features closely related to (but distinctly different from) the more familiar features of the classical groups. This paper centers on the geometries in $\mathbf{E}^7$ and $\mathbf{E}^8$ whose groups of symmetries are $G_2 \subseteq SO(7)$ and $\mathrm{Spin}(7) \subseteq SO(8)$. Both of these groups are related to the octonians (sometimes called Cayley numbers) and may be defined in terms of octonionic multiplication. In particular, G_2, the compact exceptional group of (real) dimension 14, is the subgroup of algebra automorphisms of $\mathbf{O}$ (the octonians) and $\mathrm{Spin}(7) \subseteq SO(8)$ may be defined as the subgroup of $GL_{\mathbf{R}}(\mathbf{O})$ generated by right multiplication by unit octonians which are purely imaginary.

The geometry of the algebra $\mathbf{O}$ is closely related to the complex numbers. In §1, we develop some of the properties of $\mathbf{O}$ that we need for later sections. (Our presentation is essentially borrowed from Appendix A of [12], but any mistakes are, of course, due to the author.) A particularly interesting property is described as follows: If we let $\mathrm{Im}\,\mathbf{O} \subseteq \mathbf{O}$ be the hyperplane (through 0) orthogonal to $1 \in \mathbf{O}$, and we let $S^6 \subseteq \mathrm{Im}\,\mathbf{O}$ be the space of unit vectors, then right multiplication by $u \in S^6$ induces a linear transformation $R_u\colon \mathbf{O} \to \mathbf{O}$ which is orthogonal and satisfies $(R_u)^2 = -1$. Thus, associated to each $u \in S^6$ is a complex structure on $\mathbf{O}$ (induced by $J = R_u$) which is compatible with the natural inner product on $\mathbf{O}$. We denote by $\mathbf{O}_u$ the Hermitian vector space whose underlying real vector space (with inner product) is $\mathbf{O}$ and whose complex structure is given by R_u.

Classically, this observation was used to define an almost complex structure on S^6 as follows: If $u \in S^6$, then R_u preserves the 2-plane spanned by 1 and u and therefore preserves its orthogonal 6-plane, which may be identified with $T_uS^6 \subseteq \mathrm{Im}\,\mathbf{O}$ after translation to the origin. Thus R_u induces a complex

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