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ON THE EXISTENCE OF INFINITELY MANY ISOMETRY-INVARIANT GEODESICS

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Among various problems about geodesics, one of the most prominent questions is whether or not there exist infinitely many closed geodesics on an arbitrary compact Riemannian manifold. The main interest lies in the question of whether it is possible to estimate the number of closed geodesics in terms of topological properties of the manifold only. In 1969 Gromoll and Meyer succeeded to find such a criterion [4]. They obtained the following result:

Theorem. *Let M be a compact and simply connected Riemannian manifold. Then M has infinitely many closed geodesics if the sequence of the Betti numbers for the space, with the compact-open topology, of all continuous maps $S^1 \to M$ is unbounded.*

Let us note that the spheres do not satisfy the above topological condition, though clearly the standard one has infinitely many closed geodesics. About ten years after their proof W. Klingenberg published a lecture note [10] which offers a proof of the claim that there exist infinitely many closed geodesics on any compact Riemannian manifold with finite fundamental groups. However the proof seems to need much improvement. In the same spirit as the problem of closed geodesics, one might ask if there are topological restrictions that ensure the existence of infinitely many isometry-invariant goedesics. Here a geodesic $c: R \to M$ is said to be invariant under an isometry A on M (or A-invariant) if $A(c(t)) = c(t + 1)$ for all real t. Letting id_M be the identity map on M, an id_M-invariant geodesic is simply a closed geodesic of period 1 and vice versa. Thus the theory of isometry-invariant geodesics contains that of closed geodesics. There are however some essential differences. For example a rotation on a flat torus has no invariant geodesic, although any compact Riemannian manifold has at least one closed geodesic (see e.g. [5]). While there are infinitely many closed geodesics on the standard sphere, a rotation on it has only finitely many invariant geodesics. Therefore it would be very reasonable to ask if there are infinitely many A-invariant geodesics under the

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