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FINITE PROPAGATION SPEED, KERNEL ESTIMATES FOR FUNCTIONS OF THE LAPLACE OPERATOR, AND THE GEOMETRY OF COMPLETE RIEMANNIAN MANIFOLDS

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0. Introduction

Let M^n be a complete Riemannian manifold. Then the Laplacian $\Delta = -\delta d$ on functions is a nonpositive essentially self adjoint operator when restricted to functions of compact support. Thus functions $f(\sqrt{-\Delta})$ can be defined by the spectral theorem for unbounded self adjoint operators, according to the prescription

$$f(\sqrt{-\Delta}) = \int_0^\infty f(\lambda)\, dE_\lambda, \tag{0.1}$$

where dE_λ is the projection valued measure associated with $\sqrt{-\Delta}$.

A natural problem is to study the behavior of the explicit kernel $k_{f(\lambda)}(x_1, x_2)$ representing $f(\sqrt{-\Delta})$, in terms of the behavior of various geometric quantities on M^n. As a particularly important example we have the heat kernel $E(x_1, x_2, t) = k_{e^{-\lambda^2 t}}$. By use of the local parametrix and the standard elliptic estimates, one can show that for $t > 0$, $E(x_1, x_2, t)$ is a positive (symmetric) C^∞ function of x_1, x_2, t which for fixed t and (say) x_2, is in the domain of all positive powers of Δ as a function of x_1; see e.g. [9]. In works of Gårding [19] and Donnelly [16], upper estimates for $E(x_1, x_2, t)$ (and its derivatives) were given under the assumption that M^n has bounded geometry. They showed that as $x_2 \to \infty$, the behavior of $E(x_1, x_2, t)$ is roughly similar to that of the Euclidean heat kernel, $\dfrac{e^{-\rho^2(x_1, x_2)/4}}{(4\pi t)^{n/2}}$; ($\rho(x_1, x_2)$ denotes distance). Recall that M^n is said to have bounded geometry if the injectivity radius $i(x)$ of the

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