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ALMOST FLAT MANIFOLDS

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1. Introduction

A compact riemannian manifold M is said to be ε-flat if the riemannian sectional curvature K and the diameter d of M satisfy the inequality $|K| d^2 \leqslant \varepsilon$. In [3] Gromov proved that any sufficiently flat riemannian manifold possesses a finite cover which is diffeomorphic to a nil-manifold. In addition, Gromov demonstrated that every nil-manifold carries an ε-flat metric for any $\varepsilon > 0$. Recently, following Gromov's original ideas and improving the estimate for the pinching constant $\varepsilon = \varepsilon(n)$, Buser and Karcher [2] gave a detailed proof of this result.

In the present paper we prove a stronger version of this theorem by showing that M itself, and not only a finite cover, possesses a locally homogeneous structure. In fact, we prove that under suitable curvature assumptions, M is diffeomorphic to the quotient of a simply connected nilpotent Lie group N by an affine finite extension Γ of a lattice in N. In this form the theorem generalizes the well known Bieberbach theorem on euclidean space-forms.

The main idea in this proof is the same as in Min-Oo and Ruh [9], [10], where we solved a certain partial differential equation on M. Here the additional problem stems from the fact that the elliptic operator in question is not strictly positive. The cokernel of this operator is responsible for the fact that the model N is not known a priori, but has to be constructed in the proof.

In addition to the deformation techniques employed in [9], [10], we utilize one of the main ideas in Gromov's proof, the imitation of the proof of Bieberbach's theorem as presented in Chapter 3 of Buser-Karcher [2]. However, dependence on this chapter makes estimates for the pinching constant $\varepsilon = \varepsilon(n)$ better than $\exp(-\exp n^2)$ impossible. Nevertheless, we feel that a realistic constant would be $\exp(-n^2)$; accordingly no estimate of $\varepsilon(n)$, $n = \dim M$, is given in this paper.

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