J. DIFFERENTIAL GEOMETRY
16 (1981) 739–744

# MINIMAL SETS OF FAMILIES OF VECTOR FIELDS ON COMPACT SURFACES

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## 1. Introduction

Let $M$ be a compact connected smooth manifold of dimension two, and consider a subgroup $G$ of the group of diffeomorphisms of $M$. A set $\Omega \subset M$ is $G$-invariant if $g\Omega \subset \Omega$ for all $g$ in $G$. A set is said to be $G$-minimal if it is closed $G$-invariant nonempty, and contains no such proper subset. Let $D$ be a set of smooth vector fields on $M$, and consider the group $G_D$ generated by the one-parameter group whose infinitesimal generators are the elements of $D$. When $D$ contains exactly one vector field, a well-known theorem of Schwartz [5] shows that a $G_D$-minimal set is either a point, a homeomorph of $S^1$ or all of $M$ (in the last case $M$ must be homeomorphic to a torus $T^2$). The purpose of this paper is to extend this result to arbitrary families of vector fields.

**Theorem 1.** *Let $M$ be a compact connected two-dimensional smooth manifold. Let $D$ be a set of smooth vector fields on $M$, and consider a $G_D$-minimal set $\Omega \subset M$. Then $\Omega$ must be one of the following*:

(a) *a point which is a common zero of the vector fields of $D$*;

(b) *a $G_D$-orbit homeomorphic to $S^1$*;

(c) *all of $M$.*

*Proof.* Let $m \in \Omega$, and denote by $\gamma(m)$ the $G_D$-orbit of $m$, i.e., the set of points of the form $g(m)$, $g \in G_D$. By a theorem of Sussmann [7], $\gamma(m)$ is a smooth connected paracompact submanifold of $M$ (with a natural differentiable structure) of dimension $k$, $0 \leqslant k \leqslant 2$. All vector fields in $D$ are tangent to $\gamma(m)$. If $k = 0$, $\gamma(m)$ is a point and we have (a). If $k = 2$, $\gamma(m)$ is open in $M$. Then $\overline{\gamma(m)} \setminus \gamma(m)$ is a closed invariant proper subset of $\Omega$, so $\overline{\gamma(m)} = \gamma(m) = \Omega = M$. This gives (c). If $k = 1$, $\gamma(m)$ is homeomorphic to $S^1$ or $\mathbf{R}$. In the first case we get (b). Assume that $\gamma(m)$ is homeomorphic to $\mathbf{R}$, and consider $\overline{\gamma(m)} = \Omega$. If the interior of $\Omega$ is nonempty, we conclude as before that $\Omega = M$. The theorem will be proved if we show that $\Omega$ cannot be nowhere dense when $\gamma(m)$ is homeomorphic to $\mathbf{R}$. Let us reason by contradiction, and assume that $\Omega$ is nowhere dense.

Communicated by R. Bott, April 10, 1981. The author was partially supported by $CNP_q$, Brazil.