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DIVISION ALGEBRAS AND FIBRATIONS OF SPHERES BY GREAT SPHERES

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Dedicated to Professor Buchin Su on his 80th birthday

Smooth fibrations of spheres by *great spheres* occur naturally in the study of the Blaschke conjecture. In fact, if M is a Blaschke manifold, m is a point of M, T_mM is the tangent space of M at m, $\exp_m: T_mM \to M$ is the exponential map at m, and $\mathrm{Cut}(m)$ is the cut locus of m in M, then $\exp_m^{-1}(\mathrm{Cut}(m))$ is a sphere S_m in T_mM of center 0, and $\exp_m: S_m \to \mathrm{Cut}(m)$ is a smooth *great sphere fibration* of the sphere S_m. For general information of the Blaschke conjecture, see [2].

If $\mathbf{K}$ is the real, complex, quaternionic or Cayley algebra, n is the dimension of $\mathbf{K}$ as a euclidean space, which is $1, 2, 4$ or 8, and S^{2n-1} is the unit $(2n-1)$-sphere in the euclidean $2n$-space $\mathbf{K} \times \mathbf{K}$, then there is a natural smooth great $(n-1)$-sphere fibration of S^{2n-1} such that any $(u, w), (u', w') \in S^{2n-1}$ belong to the same fibre iff either $w = w' = 0$ or $uw^{-1} = u'w'^{-1}$. When $n > 1$, this fibration, as well as isomorphic ones, is often referred as the *Hopf fibration*. Related to this result, Adams' theorem [1] says that a smooth fibration of S^{2n-1} by $(n-1)$-spheres can occur only when $n = 1, 2, 4$ or 8, and a classical theorem of Hurwitz [4] says that any division algebra $\mathbf{K}$, which possesses a norm such that for any $v, w \in \mathbf{K}$, $|vw| = |v||w|$, must be the real, complex, quaternionic or Cayley algebra. If $n = 1$ or 2, then any n-dimensional division algebra is the real or complex algebra, and any fibration of S^{2n-1} by $(n-1)$-spheres is unique up to an isomorphism. Hence in these cases, the correspondence between n-dimensional division algebras and smooth great $(n-1)$-sphere fibrations of S^{2n-1} is trivial.

In this paper, we show that for $n = 4$ or 8, each n-dimensional division algebra $\mathbf{K}$ determines a smooth great $(n-1)$-sphere fibration of S^{2n-1}, and every smooth great $(n-1)$-sphere fibration of S^{2n-1}, up to an isomorphism, is determined by an n-dimensional division algebra $\mathbf{K}$. However, it is possible

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