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FAMILIES OF PERIODIC ORBITS: LOCAL CONTINUABILITY DOES NOT IMPLY GLOBAL CONTINUABILITY

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1. Introduction

For fixed points of zeroes of a map depending on a parameter, local continuability is closely related to global continuability. Let $F: \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be a C^1 function depending on a scalar parameter α. If $F(\alpha_0, x_0) = 0$, and $D_{(\alpha, x)}F(\alpha_0, x_0)$ has full rank, then the zero (α_0, x_0) is locally continuable in the sense that a path of zeroes extends from it through a neighborhood of (α_0, x_0). The global behavior of a connected component C of zeroes through (α_0, x_0) can also be described. We have two possibilities:

(a) $C - \{(\alpha_0, x_0)\}$ is connected; or

(b) both components of $C - \{(\alpha_0, x_0)\}$ are unbounded in (α, x)-space.

It is reasonable to say that the set of zeroes through (α_0, x_0) is *globally continuable* whenever C satisfies (a) or (b). The fact that these are the only possibilities is easily seen in the generic case (where $D_{(\alpha, x)}F(\alpha, x)$ has full rank whenever $F(\alpha, x) = 0$); it has also been shown to be true in the nongeneric case, assuming only that $D_X F(\alpha_0, x_0)$ is nonsingular [1]. Hence the conditions for local continuability in fact imply global continuability.

For solutions of a differential equation $dx/dt = F(\alpha, x)$, (again depending on a parameter α), we can relate the behavior of periodic orbits to that of fixed points. Each point on a periodic orbit is a fixed point of the Poincaré return map T (to be defined later) associated with the orbit at that point. (In the following, orbit will always mean periodic orbit.) Such an orbit is locally

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