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MONOTONICITY OF INTEGRAL GAUSS CURVATURE

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1. The results

1.1. Spaces, surfaces, curves, boundaries of domains and so on are supposed to be of class C^∞ unless otherwise stated or subtended. A set S in a Riemannian space will be said to be convex if for any two points of S there exists a geodesic in S not longer than any other arc in S between the two points.

A Riemannian space M will be said to be *enlarging* (*reducing*) if for any two convex compact domains D_0 and $D_1 \subset M$ homeomorphic to a ball, the integral Gauss curvatures G_0 and G_1 of their boundaries with respect to the interior normals satisfy $G_0 \leqslant G_1$ ($G_0 \geqslant G_1$) when $D_0 \subset D_1$. (By Gauss curvature we mean product of principal normal curvatures.)

A space M either enlarging or reducing will be said to be *monotonic*.

1.2. For dimension $n = 2$, Gauss-Bonnet theorem yields a clear idea about monotonic space: those of nonpositive (nonnegative) curvatures are enlarging (reducing) and those of alternating curvature are not monotonic. Moreover, for $n = 2$, the requirement of convexity and homeomorphism to a ball can be omitted in the definition above.

We study in this paper to what extent this situation survives for $n \geqslant 3$.

One can hardly expect that a space M can be enlarging (reducing) if it contains a point p and a 2-dimensional direction σ at p where the sectional curvature is positive (negative). If for example the geodesics emanating from p and tangent to σ form a geodesic 2-dimensional surface S in a neighborhood of p, then for any distinct compact convex domains $D_0 \subset D_1 \subset S$ close to p and homeomorphic to a circle, the total curvatures C_0 and C_1 of their boundaries satisfy $C_0 > C_1$ ($C_0 < C_1$). But D_0 and D_1 can be treated as (degenerate) convex domains in M described in §1.1. Then $C_0 > C_1$ implies $G_0 > G_1$, and $C_0 < C_1$ implies $G_0 < G_1$.

For this reason, we consider only spaces of nonpositive (nonnegative) curvature for the purpose of studying enlarging (reducing) spaces. In contrast

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