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MALCEV'S COMPLETION OF A GROUP AND DIFFERENTIAL FORMS

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1. Let G be a finitely generated group, and let $G_2 = (G, G)$ be the normal subgroup of G generated by the commutators $(a, b) = a^{-1}b^{-1}ab$; $a, b \in G$. Inductively we have the sequence of normal subgroups $G_{k+1} = (G, G_k)$, $k = 1, 2, \ldots, G_1 = G$ of G and the corresponding tower of nilpotent groups $G/G_2 \leftarrow G/G_3 \leftarrow \cdots$. We assume that none of the groups G/G_k has an element of finite order. Then we talk about the group G without torsion.

A group $\mathcal{G}$ is said to be complete if for any positive integer n and any element $g \in \mathcal{G}$ the equation $x^n = g$ has at least one solution in $\mathcal{G}$. For any finitely generated nilpotent group N without torsion Malcev [4] constructed a complete nilpotent group $\overline{N}$ without torsion, called the completion of N, and an injection of N into $\overline{N}$. Furthermore he constructed a Lie algebra LN over the rationals and proved that there is a 1-1 correspondence between the complete nilpotent groups without torsion and rational Lie algebras. Thus for any finitely generated group G without torsion we have the tower of Malcev's completions

$$\overline{G/G_2} \leftarrow \overline{G/G_3} \leftarrow \cdots$$

and the tower of nilpotent rational Lie algebras

$$LG/G_2 \leftarrow LG/G_3 \leftarrow \cdots,$$

given by Malcev's theory. We talk about the Lie algebra LG of the group G. Each Lie algebra LG/G_k can be given a structure of a group by the Campbell-Hausdorff formula

$$x \circ y = x + y + \tfrac{1}{2}[x, y] + \cdots.$$

This group is isomorphic with $\overline{G/G_k}$.

On the other hand the rational homotopy type of the Eilenberg-McLane space $K(G, 1)$ is completely determined by a differential graded algebra which is free with a decomposable differential and is constructed inductively by the elementary extensions. Such algebras are said to be minimal by

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