J. DIFFERENTIAL GEOMETRY
15 (1980) 365–380

ABSTRACT WEINGARTEN SURFACES

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1. Introduction

Suppose a pair of real quadratic forms A and B is prescribed on an oriented surface S. If A is definite, one can imitate many classical procedures involving the fundamental forms I and II on a surface in 3-space. In particular, there are obvious analogs $H = H(A, B)$ and $K = K(A, B)$ of mean and extrinsic curvature, and one easily defines sequences of fundamental forms $X_n = X_n(A, B)$ and skew fundamental forms $X_n' = X_n'(A, B)$ (See §2.) If some nontrivial equation is satisfied on S connecting H, K, and (perhaps) the intrinsic curvatures of certain of the forms X_n or X_n', we call S an abstract Weingarten surface.

It is no surprise that various results from the theory of immersed surfaces can be recaptured in this setting. Indeed, a good deal of literature is based, to one extent or another, on this realization. References [7], [8], [25], [26], [27] and [28] provide just a few examples.

In this paper, we give abstract versions of some simple theorems from surface theory. In particular, we study the situation in which H and K satisfy a linear equation. We describe the exact connection between the Codazzi-Mainardi equations and the appearance in seemingly unrelated situations (see Examples 1 through 5 in §3) of certain holomorphic quadratic differentials. The Main Lemma is independent of the Codazzi-Mainardi equations, and gives information deduced from the one assumption that a particular quadratic differential associated with B is holomorphic on the conformal structure determined by A.

The usefulness of most results below depends upon the identification in natural geometric settings of pairs A, B which satisfy their hypothesis. Such applications are provided by Theorems 1 and 2, the Corollary to Theorem 3, and Example 3. In addition, the Main Lemma, Corollary 2 to Lemma 1, and Theorem 3 have already proved valuable in the study of harmonically immersed surfaces. (See [19], [20] and [22].)

Received June 19, 1978, and, in revised form, March 26, 1980.