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THE CONE TOPOLOGY ON A MANIFOLD WITHOUT FOCAL POINTS

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Introduction

Let M be a complete, simply connected Riemannian manifold without focal points. Let $\alpha(t)$ and $\beta(t)$, $t \geqslant 0$, be geodesic rays parametrized by their arc lengths, respectively. Then α and β are asymptotic if the distance between $\alpha(t)$ and $\beta(t)$ is bounded for all $t \geqslant 0$. Let $M(\infty)$ be the set of all classes of asymptotic geodesic rays and let $\overline{M} = M \cup M(\infty)$. In [4] it was proved that for any point p in M and a geodesic ray α, there exists a unique geodesic ray β asymptotic to α with $\beta(0) = p$.

Let E be $\mathbf{R}^{n+1}$ with the natural euclidean metric. Then E is an example of M. In this case two geodesic rays $\alpha(t) = a + tv(\|v\| = 1)$ and $\beta(t) = b + tw(\|w\| = 1)$ are asymptotic if and only if they are parallel, i.e., $v = w$. We denote the asymptotic class containing α by ∞v, and suppose that the ray is extended to the interval $[0, \infty]$ by putting $\alpha(\infty) = \infty v$. Then $E(\infty)$ has the natural topology as the unit sphere S^n, and $\overline{E}$ can be identified with the closed unit $(n + 1)$ − disk.

The purpose of this note is to prove the following:

Theorem. *Let M be a complete, simply connected Riemannian manifold without focal points. Then $\overline{M}$ has a canonical topology with the following property: For any $p \in M$, the exponential map: $T_pM \to M$ extends uniquely to a homeomorphism from $\overline{T_pM}$ onto $\overline{M}$.*

The topology is called the *cone topology* since for each point x in $M(\infty)$, cones containing x form a local basis at x.

The theorem is known in the case of nonpositive curvature (see [2]). In the case of no focal points, it was proved if either the dimension of M is 2, or the geodesic flow of M is of Anosov type (see [4]). The proof here refers to [3] and [4].

Proof of the theorem. Let $K(t)$ be a symmetric $n \times n$ matrix valued continuous function defined for all $t \in \mathbf{R}$, and consider the $n \times n$ matrix

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