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A SINGULAR MAP OF A CUBE ONTO A SQUARE

R. KAUFMAN

An example is given of a transformation F of class C^1 on a cube in R^3, *of rank at most* 1 *everywhere*, onto a square. With merely verbal changes, the example operates from R^{n+1} to R^n for $n = 3, 4, 5 \ldots$. The construction begins with a Cantor set $C(\beta)$ in R^3; $C(\beta)$ can be found by the standard method but the one outlined in the first paragraph leads quickly to a system of boundaries $B(n_1, \ldots, n_k)$, the main geometrical curiosity of this example.

We learned of this kind of problem from Kevin Grasse and Felix Albrecht; it was stated by M. Hirsch (*Differential topology*, Graduate Texts in Math. Vol. 33, Springer, Berlin, 1976, p. 74).

A system of cubes. For each number β in $(0, 1/2)$ we define a method of constructing 8 subcubes in each cube in R^3. Let the larger cube be defined by the inequalities $|x_i - c_i| \leqslant L/2$, $1 \leqslant i \leqslant 3$. Then the subcubes are defined by $|x_i - c_i \pm L/4| \leqslant \beta L/2$, so there are 8 in all; any two have a distance $\geqslant L/2 - \beta L$, and all have a distance $\geqslant L/4 - \beta L/2$ from the boundary of the large cube.

Beginning with the cube I_0: $|x_i| \leqslant 1$, we define cubes $I(n_1, \ldots, n_k)$, wherein $I(n_1)$ are the 8 cubes obtained from I_0, etc., and each $n_k = 1, 2, \ldots, 8$. Distinct cubes $I(n_1, \ldots, n_k)$ and $I(n'_1, \ldots, n'_k)$ have a distance at least $\beta^{k-1} - 2\beta^k$. In the case of cubes $I(n_1, \ldots, n_k)$ and $I(n'_1, \ldots, n'_j)$ with $k \leqslant j$, the situation is more complicated. When the cubes are disjoint we have the lower bound $\beta^{k-1}(1 - 2\beta)$ found above; when the larger contains the smaller, the distance between their boundaries exceeds $\beta^k(1/2 - \beta)$. We denote the boundary of I_0 by B_0, and the boundary of $I(n_1, \ldots, n_k)$ by $B(n_1, \ldots, n_k)$. The Cantor set defined by the cubes is denoted $C(\beta)$ and we require $\beta^2 > 1/8$, for reasons to appear presently.

A mapping of $C(\beta)$. Let R_0 be any closed cube in R^2, and the rectangles $R(n_1, \ldots, n_k)$ be defined by this variant of the process used above. When k is even (or $R = R_0$) we divide $R(n_1, \ldots, n_k)$ by 7 vertical lines into 8 congruent rectangles; when k is odd we divide by horizontal lines. Thus $R(n_1, \ldots, n_k)$ has diameter $\leqslant C2^{-3k/2}$.

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