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A GLOBAL VERSION OF THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

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1. Introduction

In [4], Tonti gave necessary and sufficient conditions for certain differential expressions (namely those expressions which we call "source equations"; for the definition see below or [3]) to be *locally* the Euler equation of some variational problem. In this paper we consider the corresponding global problem. To state the results we need some definitions.

In what follows, $\pi: E \to W$ will be some fixed differentiable fibration i.e., (E, π, W) is a fibre bundle, E and W are smooth manifolds and π has everywhere maximal rank. A *variational problem*, or *Lagrangian*, on π is an operator $\mathfrak{L}$ which assigns to each smooth (local) section S: $W \to E$ of π an n-form ($n = \dim(W)$) $\mathfrak{L}(S)$ on the domain of S such that, for each x in the domain of S, $(\mathfrak{L}(S))(x)$ only depends (smoothly) on the value of S, and on a finite number of its derivatives, in x.

A *source equation* on a bundle π is an operator $\mathfrak{S}$ which assigns to each smooth (local) section S: $W \to E$ of π and each x in the domain of S an element $(\mathfrak{S}(S))(x) \in (\operatorname{Ker}(d\pi)_{S(x)})^* \otimes \Lambda^n(T_x^*(W))$, which only depends (smoothly) on the value of S, and a finite number of its derivatives in x.

A source equation $\mathfrak{S}$ is the Euler equation of the Lagrangian $\mathfrak{L}$ if for each bounded (i.e., having compact closure) oriented open $U \subset W$, and each smooth 1-parameter family of local sections S_t of π with the properties:

(i) for each $t \in (-\varepsilon, +\varepsilon)$, $\overline{U} \subset$ interior of the domain of S_t, and

(ii) $S_t(x)$ is independent of t if $x \notin U$, we have

$$\frac{d}{dt}\int_{\overline{U}} \mathfrak{L}(S_t)\bigg|_{t=0} = \int_{\overline{U}} \langle \mathfrak{S}(S_t)(x), \dot{S}_t(x)\rangle\bigg|_{t=0},$$

where $\dot{S}_t(x)$ denotes the tangent vector of the curve $t \to S_t(x)$; this tangent vector is in $\operatorname{Ker}(d\pi)_{S_t(x)}$ so for each x, $\langle \mathfrak{S}(S_t)(x), \dot{S}_t(x)\rangle|_{t=0} \in \Lambda^n(T_x^*(W))$. Hence on both left- and right-hand side there is an n-form under the integral sign. The integral is defined because U is oriented and bounded.

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