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THE HEAT EQUATION AND MODULAR FORMS

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1. Introduction

Let G be a compact semi-simple simply connected Lie group, and let Δ be the Casimir operator on G. Then the heat equation on G is $\Delta u + \partial u/\partial t = 0$. We denote by $H(x, -t/2\pi i)$ the fundamental solution of the heat equation, and by $Z(t)$ the trace of the heat Kernel. The aim of this paper is to investigate how $H(x, t)$ behaves under the transformation $t \to -1/t$. Two main conclusions of this investigation are the following results.

Theorem 1.1. *$e^{i\pi kt/12}H(a, t) = \eta(t)^k$, where $k = \dim G$, η is the Dedekind η-function, that is, $\eta(t) = e^{i\pi t/12} \prod_{n=1}^{\infty} (1 - e^{2\pi int})$, and a is an element of G which is principal of type ρ. (For a definition of elements principal of type ρ see* [5] *or* § 5 *of this paper.)*

Theorem 1.2. *$Z(t) \sim (4\pi t)^{-(1/2)k} \operatorname{vol} G \exp(kt/24)$ is the asymptotic expansion for small t of the trace of the heat kernel.*

The first of these two results is a form of Macdonald's η-function identities. These identities can be written (see [6])

$$e^{\pi ikt/12} \Sigma d(\lambda + \rho)e^{2\pi ic(\lambda)t} = \eta(t)^k , \tag{1.1}$$

where the summation is over a suitable lattice which together with the other notation will be explained later. In [5] Kostant observed that this could be written as a sum over the dominant weights with a suitable weighting factor $\varepsilon(\lambda)$ as

$$e^{\pi ikt/12} \Sigma\varepsilon(\lambda)d(\lambda + \rho)e^{2\pi ic(\lambda)t} = \eta(t)_k . \tag{1.2}$$

Kostant identified the weighting $\varepsilon(\lambda)$ as the value of the character with highest weight λ at the point a which is principal of type ρ, that is, $\varepsilon(\lambda) = \chi_\lambda(a)$. With this result (1.2) can be interpreted in terms of the fundamental solution of the heat equation. In fact Theorem 1.1 is such an interpretation. However, our proof of Theorem 1.1 is independent of these two previous results, given by (1.1) and (1.2). Thus there is now a set of three results any two of which imply the third.

The result of Theorem 1.2 was first obtained by McKean and Singer [7] in the case of the group S^3. More recently Urakawa [10] has obtained this result

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