J. DIFFERENTIAL GEOMETRY
13 (1978) 361-371

THE POINCARÉ LEMMA FOR $d\omega = F(x, \omega)$

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Introduction

One can locally solve the equation $d\alpha = \beta$ only if the $(p + 1)$-form β satisfies $d\beta = 0$. Poincaré's lemma states that this condition is also sufficient. We wish to consider a nonlinear version of this and to relate it to the Frobenius theorem. This theorem, in its classical formulation concerns a system of partial differential equations $\partial f_i/\partial x_j = F_{ij}(x, f)$ and asserts that one can solve these equations and have f take on a given value at any point in a region if and only if $\partial F_{ij}/\partial x_k = \partial F_{ik}/\partial x_j$ when the derivatives $\partial f_i/\partial x_j$ which occur in the use of the chain rule are replaced by $F_{ij}(x, f)$. In this paper we consider the equation $d\omega = F(x, \omega)$ where ω is a p-form, and $F(x, \omega)$ a $(p + 1)$-form. We discuss the analogue of the condition of Frobenius and show it is a sufficient condition for local solvability (§ 1). Both the Poincaré lemma and the Frobenius theorem are included in our formulation. In § 2 we consider various geometric applications. Finally in the last section we return to the analogy with the Frobenius theorem and show that our sufficient condition is also necessary for the existence of solutions to a certain special initial value problem.

It might be interesting to try to obtain our results, in the real analytic case, using the Cartan-Kaehler theorem. Note the proposition in § 1 does not quite state that $\{d\omega - F(\omega), dF(\omega)\}$ generates under the wedge product a differential ideal, when the coefficients of ω are admitted as new independent variables. However, it may be that various generalizations of the Cartan-Kaehler theorem, for instance the work of Goldschmit [3], do include as a very special case the present results for real analytic data. See also our comments in § 2 on a paper by Gasqui.

All our discussion will be local. Let M^N be an open subset of $\boldsymbol{R}^N$. Let Λ_x^p be the space of p-forms at the point $x \in M^N$, $\Lambda_x^p T(M^N)$ the dual space of p-vectors, and Γ_x^p the space of germs of p-forms at x. Recall that forms α and β define the same element in Γ_x^p if there is some open neighborhood U of x with $\alpha = \beta$ on U. For sanity's sake, we identify α with the element in Γ_x^p which it defines. Often we delete the base point x. We will sometimes consider submanifolds $M \subset M^N$ and the associated spaces $\Lambda_x^p(M)$ and $\Gamma_x^p(M)$. We introduce the germs in order to avoid specifying on what neighborhood of a given point each of

Received November 2, 1976, and, in revised form, June 10, 1977. This work supported in part by NSF grants MPS 75-05577 and MCS 76-06978.