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MANIFOLDS WITHOUT FOCAL POINTS

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0. Introduction

The behavior of geodesics in Riemannian manifolds without conjugate or focal points has been discussed by many geometers such as Morse, Hedlund, Green, Eberlein and others. It is known that the properties, e.g., the topological transitivity of geodesic flows on certain Riemannian manifolds, are connected closely to some instability property of geodesics. On a complete simply connected Riemannian manifold without conjugate points, L. Green proved an instability property of geodesics under the condition that the sectional curvature is bounded from below. His proof for the higher dimensional case was incomplete as was pointed out by Eberlein.

The purpose of this paper is to extend the theory of L. Green in [6], [7] and [8], reproducing the results there without the condition on curvature assumed by Green. Consequently our results turn out to be extensions of some fundamental notions and results for nonpositively curved manifolds to manifolds without focal points.

A complete Riemannian manifold M is said to have *no focal points* if no maximal geodesic σ of M has focal points along any perpendicular geodesic, where σ is considered as an imbedded one-dimensional submanifold of M. This property can be stated as follows: For any geodesic ray γ and any nontrivial Jacobi field along γ vanishing at $t = 0$, $(d/dt)\langle Y(t), Y(t)\rangle > 0$ for $t > 0$, where $\langle\ ,\ \rangle$ denotes the inner product with respect to the Riemannian metric of M, see [12].

In this paper we shall be concerned only with Riemannian manifolds without focal points. In addition manifolds are always assumed to be connected, complete and differentiable (of class C^∞). Geodesics are assumed to have unit speed unless otherwise stated.

In § 1 we introduce the basic results on the Jacobi equations by L. Green for later use.

In § 2 we prove

Theorem 1. *Let M be a complete Riemannian manifold without focal points. Let γ be a geodesic ray with $\gamma(0) = p \in M$. If $Y^x(t)$ denotes the Jacobi field along γ with $Y^x(0) = 0$, $(Y^x)'(0) = x$, where x is nonzero vector at p, then*

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