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THE HOPF-RINOW THEOREM IN INFINITE DIMENSION

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I. Statement of results

We begin by reviewing some essential features. By a *Riemannian manifold* M we understand a connected C^∞-manifold modelled on some Hilbert space H, such that the tangent space $TM_p \simeq H$ carries a scalar product $\langle \cdot, \cdot \rangle_p$ which is C^∞ in $p \in M$ and defines on TM_p a norm $\|\cdot\|_p$ equivalent to the original norm of H.

If p and q are two points in M, a *path* from p to q is a continuous map $c: [0, 1] \to M$ such that $c(0) = p$ and $c(1) = q$. The set of all piecewise C^∞ paths from p to q will be denoted by $\mathscr{C}_p^q$. If $c \in \mathscr{C}_p^q$ is such a path, its *length* $L_p^q(c)$ is the real number defined by

$$L_p^q(c) = \int_0^1 \|\dot{c}(t)\|_{c(t)} dt . \tag{1.1}$$

The *geodesic distance* d on M is defined by

$$d(p, q) = \inf\{L_p^q(c) \mid c \in \mathscr{C}_p^q\} , \qquad \forall p, q \in M . \tag{1.2}$$

It is compatible with the manifold topology of M. Any path $c \in \mathscr{C}_p^q$ such that $d(p, q) = L_p^q(c)$ and the speed $\|\dot{c}\|_c$ is constant will be called a *minimal geodesic*; it must be C^∞ and satisfy the equation (where ∇ denotes the Levi-Civita connection)

$$\nabla_{\dot{c}(t)}\dot{c}(t) = 0 , \tag{1.3}$$

which means that $\dot{c}(t)$ is obtained from $\dot{c}(0) \in TM_p$ by parallel translation along c. Conversely, any solution c of (1.3) is called a geodesic. The manifold M will often be assumed to be complete for the metric d; this will imply that solutions of (1.3) are defined for all $t \in \boldsymbol{R}$, i.e., that geodesics can be indefinitely extended.

Throughout this paper, for $\delta > 0$ and $p \in M$, we shall use the following notations:

$$B_p^\delta = \{\xi \in TM_p \mid \|\xi\|_p < \delta\} , \qquad S_p^\delta = \{\xi \in TM_p \mid \|\xi\|_p = \delta\} , \tag{1.4}$$

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