J. DIFFERENTIAL GEOMETRY
13 (1978) 273-278

UMBILICAL SUBMANIFOLDS OF SASAKIAN SPACE FORMS

DAVID E. BLAIR & LIEVEN VANHECKE

1. The purpose of this note is to prove the following theorem:

Theorem. *Let N^n, $n \geq 3$, be an umbilical submanifold of a Sasakian space form $M^{2m+1}(c)$. If the mean curvature vector is parallel in the normal bundle, then N^n is one of the following:*

(i) *N^n is a real space form immersed as an integral submanifold of the contact distribution, and N^n is totally geodesic when $n = m$.*

(ii) *The characteristic vector field of the contact structure is tangent to N^n, N^n is totally geodesic and N^n is a Sasakian space form with the same ϕ-sectional curvature.*

(iii) *$c = 1$ and N^n is a real space form.*

If the mean curvature vector is not parallel, then

(iv) *N^n is an anti-invariant submanifold, and if N^n has constant mean curvature, then $c < -3$ and N^n admits a codimension* 1 *foliation by umbilical submanifolds of type* (i).

The four cases of the theorem do occur. In fact, the first three can occur in the odd-dimensional sphere $S^{2m+1}(1)$; for example $S^{2m+1}(1)$ admits a great m-sphere which is an integral submanifold of the usual contact structure [1] and a codimension 2 great sphere such that the characteristic vector field is tangent and the sphere inherits the contact structure of S^{2m+1}. Sasakian submanifolds of Sasakian manifolds have been studied quite extensively; see e.g. [2], [4]. In $\boldsymbol{R}^{2m+1}$ with coordinates (x^i, y^i, z), the usual contact form $\eta = \frac{1}{2}(dz - \sum y^i dx^i)$ together with the Riemannian metric $G = \eta \otimes \eta + \frac{1}{4} \sum ((dx^i)^2 + (dy^i)^2)$ is a Sasakian structure with constant ϕ-sectional curvature equal to -3. The vector fields $\partial/\partial y^i$ span an integrable distribution whose leaves are integral submanifolds of the contact distribution $\eta = 0$. Moreover these submanifolds are totally geodesic (see e.g. [1]) and G restricted to these submanifolds is just the Euclidean metric. Hence taking an $(n-1)$-sphere $\sum (y^i)^2 =$ constant we have an umbilical submanifold in $\boldsymbol{R}^{2m+1}(-3)$. We devote § 5 to an example of type (iv).

2. Let M be a $(2m + 1)$-dimensional contact manifold with contact form η, i.e., $\eta \wedge (d\eta)^m \neq 0$. It is well known that a contact manifold admits a vector field ξ, called the *characteristic vector field,* such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$.

Communicated by K. Yano, October 2, 1976.